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TRAFFIC GENERATED BY A SEMI-MARKOV ADDITIVE PROCESS

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We consider a semi-Markov additive process $A(\cdot)$—that is, a Markov additive process for which the sojourn times in the various states have general (rather than exponential) distributions. Letting the Lévy processes $X_i(\cdot)$, which describe the evolution of $A(\cdot)$ while the background process is in state i, be increasing, it is shown how double transforms of the type $\int_0^\infty e^{-qt}\,\mathbb{E}[e^{-\alpha A(t)}\,dt]$ can be computed. It turns out that these follow, for given nonnegative α and q, from a system of linear equations, which has a unique positive solution. Several extensions are considered as well.

1. INTRODUCTION

Markov additive processes (MAPs) have proven an important modeling tool in communications networking [13, Chaps. 6 and 7] as well as finance [4,10], whereas nowadays also applications in biology are envisaged (see, e.g., [9]). This has led to a vast body of literature; for an overview, see, for instance, [3, Chap. XI]. A MAP is essentially a Lévy process whose Laplace exponent depends on the state of a (finite-state) Markovian background process; while this background process is in state i, the MAP, say $A(\cdot)$, evolves as a Lévy process $Y_i(\cdot)$ characterized by its Lévy exponent $\varphi_i(\cdot)$ [5]. MAPs can be considered a nontrivial generalization of the standard Lévy process, and many results that were established earlier for Lévy processes were extended to the MAP framework. In particular, transforms of the type $\mathbb{E}\exp(-\alpha A(t))$, for given nonnegative α and t, can be given explicitly in terms of a matrix exponential ([3, Prop. 2.1]; compare also [11]).

In a MAP it is implicit that the sojourn times in the states of the background process are exponentially distributed. Less is known about the situation in which this exponentiality assumption is lifted. To the best of our knowledge, one of the few results known [7, Eq. (2.1.3)] is for a very special case. The situation considered in [7] corresponds to a two-state background process, where one of the states (the "off-state") corresponding to $A(\cdot)$ being constant and the other state (the "on-state") corresponds to $A(\cdot)$ growing with constant speed; the on-times are allowed to have a general distribution, but the off-times are still assumed to be exponentially distributed. The result is in terms of double transforms of the type $\int_0^\infty e^{-qt}\,\mathbb{E}[e^{-\alpha A(t)}]\,dt$.

The goal of this note is to consider the situation of general sojourn times for *all* states of the background process, in which we could call $A(\cdot)$ a *semi-Markov additive process* (sMAP). We show that for a given (nonnegative) value of α and q, the double transforms

$$\int_0^\infty e^{-qt}\,\mathbb{E}_i\left[e^{-\alpha A(t)}1_{\{X(t)=j\}}\right]dt$$

satisfy a linear system of equations, here, $X(\cdot)$ records the state of the background process and $\mathbb{E}_i(\cdot)$ denotes expectation given that the background process starts off in state i at time 0. In addition, we show that this system of linear equations has a unique positive solution. Bearing the applications in, for example, communications networking and biology in mind, we let the Lévy processes $Y_i(\cdot)$ be increasing, but we also comment on what changes if we relax this assumption.

We proceed by presenting the formal model description of an sMAP. This sMAP $A(\cdot)$ is defined as follows (where it is assumed that $A(0)=0$):

- Let X_n be a discrete-time, irreducible Markov chain living on a finite state space $I := \{1,\dots,N\}$. Its transition matrix is given by $P \equiv (p_{ij})_{i,j=1}^N$, and the corresponding invariant distribution is ϱ.
- Let $(B_{i,n})_{n\in\mathbb{N}}$ be, for any $i \in I$, a sequence of nonnegative independent and identically distributed (i.i.d.) random variables, distributed as the generic random variable B_i; the N sequences $(B_{i,n})_{n\in\mathbb{N}}$ are assumed to be mutually independent. Let $X(t)$ be a (continuous-time) semi-Markov chain on I, defined as follows. Supposing $X(0)=i$, the background process $X(\cdot)$ stays in i for a period that is distributed as B_i. Then $X(t)$ jumps according to the transition matrix P to some new state j. It stays there for a time distributed according to B_j and so forth. It is assumed that $\mathbb{E}B_i$ is finite for all i.
- While in state i, $A(\cdot)$ evolves as an increasing Lévy process (also referred to as "subordinator") $Y_i(t)$. Lévy processes are stochastic processes with stationary,

independent increments; see, for example, [5] or [3, Chap. IX]. We denote the Laplace exponent of $Y_i(t)$ by $\varphi_i(\cdot)$:

$$\mathbb{E}e^{-\alpha Y_i(t)} = e^{-\varphi_i(\alpha)t}.$$

The Lévy processes $Y_i(\cdot)$ are independent of the background process $X(\cdot)$.

We finish the model description by providing a number of examples of Lévy subordinators. An example used frequently is that of linear drifts; then $\varphi_i(\alpha) = \alpha r_i$, for $r_i \geq 0$. A second leading example is that of compound Poisson processes: While the background process is in state i, an i.i.d. sequence of jobs (distributed as a random variable J_i) arrives according to a Poisson process of rate λ_i, leading to

$$\varphi_i(\alpha) = \lambda(1 - \mathbb{E}e^{-\alpha J_i}).$$

For the case of jobs of size 1, this reduces to $\varphi_i(\alpha) = \lambda_i(1 - e^{-\alpha})$. A last example relates to the record process of a Lévy process $Z(\cdot)$. Defining $T_x := \inf\{t : Z(t) > x\}$, it is easily seen that the increasing process T_x has stationary independent increments and is therefore a Lévy process.

2. ANALYSIS

For ease, we start in our analysis by considering special increasing Lévy processes: While in state i, $A(\cdot)$ grows at a linear rate $r_i \geq 0$; later we consider the general case. In this "linear drift case", we have

$$A(t) = \int_0^t r_{X(t)}\, dt.$$

This $A(\cdot)$ is often used as input for fluid queuing models; see, for example, [2,6,12].

We are interested in the so-called double transform

$$\mathscr{K}_{ij}(\alpha, q) := \int_0^\infty e^{-qt}\, \mathbb{E}_i\left[e^{-\alpha A(t)} 1_{\{X(t)=j\}}\right] dt,$$

where it is assumed that the background process has just jumped to state i at time 0 (we come back to this issue in Remark 2.3). Interestingly, this transform can be alternatively written as

$$\frac{1}{q}\int_0^\infty qe^{-qt}\, \mathbb{E}_i\left[e^{-\alpha A(t)} 1_{\{X(t)=j\}}\right] dt = \frac{\mathscr{L}_{ij}(\alpha, q)}{q},$$

$$\mathscr{L}_{ij}(\alpha, q) := \mathbb{E}_i\left[e^{-\alpha A(\tau_q)} 1_{\{X(\tau_q)=j\}}\right],$$

where τ_q is an exponentially distributed random variable with mean $1/q$; we call τ_q the "killing epoch". We first decompose

$$\mathscr{L}_{ij}(\alpha, q) = \mathbb{E}_i\left[e^{-\alpha A(\tau_q)} 1_{\{X(\tau_q)=j, \tau_q < B_i\}}\right] + \mathbb{E}_i\left[e^{-\alpha A(\tau_q)} 1_{\{X(\tau_q)=j, \tau_q \geq B_i\}}\right];$$

these terms we call I_1 and I_2, respectively.

First, consider I_1. Then the killing epoch, τ_q, takes place before B_i (i.e., the end of the sojourn time in state i). We thus obtain

$$I_1 = \int_0^\infty \int_s^\infty q e^{-sq} e^{-\alpha r_i s} f_{B_i}(t) 1_{\{i=j\}} \, dt \, ds.$$

Changing the order of integration, we eventually obtain

$$I_1 = 1_{\{i=j\}} \frac{q}{q+\alpha r_i}(1 - L_i(q+\alpha r_i)),$$

where $L_i(\cdot)$ is the Laplace transform of B_i.

Now, we consider I_2. Observe that if the sojourn time of state i ends before the killing epoch, we can let the Markov chain jump and sample the killing epoch again, due to the memoryless property. This reasoning leads to

$$I_2 = \int_0^\infty \int_0^s q e^{-sq} e^{-\alpha r_i t} f_{B_i}(t) \, dt \, ds \left(\sum_{k \neq i} p_{ik} \mathscr{L}_{kj}(\alpha, q) \right).$$

Again interchanging the integrals, we obtain

$$I_2 = L_i(q+\alpha r_i) \left(\sum_{k \neq i} p_{ik} \mathscr{L}_{kj}(\alpha, q) \right).$$

We have thus arrived at the following result.

THEOREM 2.1: *Fix the final state j and the values of α and q (assumed nonnegative). Then the vector*

$$x \equiv (\mathscr{L}_{1j}(\alpha, q), \ldots, \mathscr{L}_{Nj}(\alpha, q))^{\mathrm{T}}$$

is the unique solution of a system of equations $Ax = b$. Here the entries of the matrix $A := I - \tilde{P}$ are given by

$$\tilde{P}_{ik} := L_i(q+\alpha r_i) p_{ik},$$

which is between 0 and 1. In addition, the vector $b \equiv (b_1, \ldots, b_N)^{\mathrm{T}}$ is given by

$$b_i := 1_{\{i=j\}} \frac{q}{q+\alpha r_i}(1 - L_i(q+\alpha r_i)).$$

The uniqueness of the solution follows from the fact that A is (strictly) diagonally dominant for nonnegative α and q and, hence, invertible.

COROLLARY 2.2: *Consider the above model, but now with the constant drifts (with slope r_i) replaced by Lévy subordinators $Y_i(\cdot)$ (with Laplace exponent $\varphi_i(\alpha)$). Then Theorem 2.1 goes through, with $q + \alpha r_i$ replaced by $q + \varphi_i(\alpha)$.*

Remark 2.3: Above we assumed that the background process had just jumped to state i at time 0. In this remark we wish to compute the double transform when the background process starts off in equilibrium at time 0. Assuming that the generic random variables B_i, with $i = 1, \dots, N$, have finite mean, it is clear that the long-run fraction of time that the background process $X(\cdot)$ spends in state i is given by

$$\pi_i := \frac{\varrho_i \mathbb{E}B_i}{\left(\sum_{j=1}^N \varrho_j\right)\mathbb{E}B_j}.$$

Our goal is then to compute

$$\mathscr{K}_{\pi,j}(\alpha, q) := \int_0^\infty e^{-qt}\, \mathbb{E}_\pi\big[e^{-\alpha A(t)} 1_{\{X(t)=j\}}\big]\, dt,$$

where the subscript π denotes that we start off in equilibrium at time 0. It follows from the theory of semi-Markov processes that the state of the background process at time 0 is distributed according to π. It is important to note, however, that supposing that this state is i, the time until the first jump is *not* distributed according to B_i, but according to its *residual lifetime* variant $B_i^\star$:

$$\mathbb{P}(B_i^\star \le x) = \frac{1}{\mathbb{E}B_i}\int_0^x \mathbb{P}(B_i > y)\, dy.$$

Let $\hat{f}_{B_i}(\cdot)$ be the density of the residual lifetime and $\hat{L}_i(\alpha)$ be the corresponding Laplace transform:

$$\hat{f}_{B_i}(x) = \frac{\mathbb{P}(B_i > x)}{\mathbb{E}B_i}, \qquad \hat{L}_i(\alpha) = \frac{1 - L_i(\alpha)}{\alpha \mathbb{E}B_i}.$$

We obtain that, with $\mathscr{L}_{\pi,j}(\alpha, q) := q\mathscr{K}_{\pi,j}(\alpha, q)$,

$$\mathscr{L}_{\pi,j}(\alpha, q) := \sum_{i=1}^N \pi_i \hat{\mathscr{L}}_{ij}(\alpha, q),$$

where $\hat{\mathscr{L}}_{ij}(\alpha, q)$ equals

$$1_{\{i=j\}} \frac{q}{q + \varphi_i(\alpha)} \left(1 - \hat{L}_i(q + \varphi_i(\alpha))\right) + \hat{L}_i(q + \varphi_i(\alpha)) \left(\sum_{k \ne i} p_{ik} \mathscr{L}_{kj}(\alpha, q)\right);$$

note that the $\mathscr{L}_{ij}(\alpha, q)$ can be computed using Theorem 2.1. These formulas match the results for the two-state case in [7, Eq. (2.1.3)].

Example 2.4: We here consider a two-state sMAP, with $p_{11} = p_{22} = 0$. Fix $j = 1$; for reasons of symmetry the results for $j = 2$ follow directly from those for $j = 1$. Denote

$$\zeta_i(\alpha, q) = L_i(q + \varphi_i(\alpha)), \qquad \eta_i(\alpha, q) := \frac{q}{q + \varphi_i(\alpha)}.$$

It follows that

$$A^{-1} = \frac{1}{1 - \zeta_1(\alpha, q)\zeta_2(\alpha, q)} \begin{pmatrix} 1 & \zeta_1(\alpha, q) \\ \zeta_2(\alpha, q) & 1 \end{pmatrix},$$
$$b = (\eta_1(\alpha, q)(1 - \zeta_1(\alpha, q)),\ 0)^{\mathrm{T}}.$$

Elementary computations yield that

$$\begin{pmatrix} \mathscr{L}_{11}(\alpha, q) \\ \mathscr{L}_{21}(\alpha, q) \end{pmatrix} = \frac{\eta_1(\alpha, q)(1 - \zeta_1(\alpha, q))}{1 - \zeta_1(\alpha, q)\zeta_2(\alpha, q)} \begin{pmatrix} 1 \\ \zeta_2(\alpha, q) \end{pmatrix}.$$

After lengthy calculations, it follows that, with $\xi_i(q) := 1/(q\,\mathbb{E}B_i)$ and suppressing the arguments of ξ_i, ζ_i, and η_i,

$$\hat{\mathscr{L}}_{11}(\alpha, q) = \eta_1 - \xi_1 \frac{\eta_1^2(1 - \zeta_1)(1 - \zeta_2)}{1 - \zeta_1\zeta_2},$$
$$\hat{\mathscr{L}}_{21}(\alpha, q) = \xi_2 \frac{\eta_1\eta_2(1 - \zeta_1)(1 - \zeta_2)}{1 - \zeta_1\zeta_2}.$$

Taking $\phi_1(\alpha) = 0$ and $\phi_2(\alpha) = \alpha$ (so that we obtain an "on–off source" that alternates between transmitting at a constant rate 1 and being silent) and assuming the off-times to have an exponential distribution, we indeed retrieve Eq. (2.1.3) of [7].

3. CONCLUDING REMARKS

The results presented in this note are in terms of double transform that, in general, cannot be inverted explicitly. Instead, one has to rely on numerical techniques to obtain accurate approximations for probabilities of the type $\mathbb{P}(A(t) > x)$. It is noted that, recently, substantial progress has been made with respect to this type of inversion techniques. In addition to the 'classical' reference [1], we wish to draw attention on novel ideas developed by den Iseger, reported in [8].

When the Lévy processes $Y_i(\cdot)$ are not necessarily subordinators, one clearly needs to work with characteristic exponents rather than Laplace exponents. One can easily derive the system of linear equations that is solved by the transform of the characteristic function of $A(t)$.

In many applications from practice (in particular, those from biology), one considers the following situation. The process $X(\cdot)$ alternates between two states—say 1 and 2—and in state i, particles are generated according to a Poisson process with rate

$\lambda_i \geq 0$. For general sojourn-time distributions, the double transform of $A(t)$ can be computed as described in this article. One could, however, assume that every particle remains in the system for an exponential time (say, with mean $1/\mu$) and suppose that the goal is to find the distribution of the number of particles $N(t)$ that is present at time t. It is clear that the theory that we developed does not apply anymore; observe that the rate of particles leaving is proportional to the number of particles present (as in the $M/G/\infty$ queue). The analysis presented in this article might serve as a first step toward finding the distribution of the number of particles $N(t)$ present at time t or the corresponding steady-state distribution $N(\infty)$.

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